

1.1 – Introduction to Systems of Linear Equations

Definition: A linear equation can be expressed in the form

$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ where a_i are constants, not all zero, and b is a constant.

#14 In each part, use parametric equations to describe the solution set of the linear equation.

a. $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

a. $x = -10y + 2$

Let $t = y$

Then

this is a parameter

$$x = -10t + 2$$

$$y = t$$

b. $x_1 = -3x_2 + 12x_3 + 3$

Let $s = x_2$, $t = x_3$

Then

$$x_1 = -3s + 12t + 3$$

$$x_2 = s$$

$$x_3 = t$$

c. $x_1 = -\frac{1}{2}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4 + 5$

Let $r = x_2$, $s = x_3$, $t = x_4$

Then

$$x_1 = -\frac{1}{2}r - \frac{3}{4}s - \frac{1}{4}t + 5$$

$$x_2 = r$$

$$x_3 = s$$

$$x_4 = t$$

Definition: A **parameter** (in this context) is a variable that when assigned a value provides a specific solution.

Definitions: A **homogeneous linear equation** is a linear equation in which $b = 0$. hom. homog.

A **system of linear equations** is a finite set of linear equations. It has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Note that the first of the double subscripts references the equation number, and the second references the term in that equation.

Definition: The **augmented matrix** for the above linear system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \text{ or } \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

#16 Each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

a. $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 6 & 2 & -8 \\ 3 & 1 & -4 \end{array} \right] \underline{R_2} \rightarrow R_2 - \frac{1}{2} R_1$

$$\begin{array}{ccc} 3 & 1 & -4 \\ -3 & -1 & 4 \\ \hline 0 & 0 & 0 \end{array} \quad \left[\begin{array}{cc|c} 6 & 2 & -8 \\ 0 & 0 & 0 \end{array} \right] \quad R_1 \rightarrow \frac{1}{2} R_1$$

$$\left[\begin{array}{cc|c} 3 & 1 & -4 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} 3x_1 + x_2 &= -4 \\ x_1 &= -\frac{1}{3}x_2 - \frac{4}{3} \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &= -\frac{1}{3}t - \frac{4}{3} \\ x_2 &= t \end{aligned}}$$

$$\begin{aligned} 2x - y + 2z &= -4 \\ \text{b. } 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 6 & -3 & 6 & -12 \\ -4 & 2 & -4 & 8 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 + 2R_1$$

$$\begin{array}{cccc} 6 & -3 & 6 & -12 \\ -6 & 3 & -6 & 12 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{cccc} -4 & 2 & -4 & 8 \\ 4 & -2 & 4 & -8 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$R_1: 2x - y + 2z = -4 \Rightarrow x = \frac{1}{2}y - z - 2$$

$$\begin{array}{c} \uparrow \quad \quad \uparrow \\ s=y \quad t=z \end{array}$$

$$\boxed{\begin{aligned} x &= \frac{1}{2}s - t - 2 \\ y &= s \\ z &= t \end{aligned}}$$

Definitions: The following are **elementary row operations** performed on a matrix.

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Definitions: A **solution** of a linear system is a sequence of n numbers $s_1, s_2, \dots, s_n$ that when substituted for corresponding unknowns x_i makes each equation a true statement. If $n = 2$, then the solution is an **ordered pair**, and if $n = 3$, it is an **ordered triple**. In general, an **ordered n-tuple** has the form $(s_1, s_2, \dots, s_n)$. A linear system is **consistent** if it has at least one solution. A linear system is **inconsistent** if it has no solutions.

#19 Find all values of k for which the given augmented matrix corresponds to a consistent linear system.

$$\begin{array}{c} x \quad y \\ \text{a.} \left[\begin{array}{cc|c} 1 & k & -4 \\ 4 & 8 & 2 \end{array} \right] \end{array}$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$\begin{array}{ccc} 1 & 8 & 2 \\ -4 & -4k & 16 \\ \hline 0 & 8-4k & 18 \end{array}$$

$$\text{b.} \left[\begin{array}{cc|c} 1 & k & -1 \\ 4 & 8 & -4 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$\begin{array}{ccc} 1 & 8 & -4 \\ -4 & -4k & 4 \\ \hline 0 & 8-4k & 0 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & k & -4 \\ 0 & 8-4k & 18 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & k & -1 \\ 0 & 8-4k & 0 \end{array} \right]$$

$$R_2: (8-4k)y = 18$$

$$R_2: (8-4k)y = 0 \quad \downarrow$$

If $k \neq 2$, the system has a solution

Has a soln for $k \in \mathbb{R}$ in

Example: A 3-7-9 diet calls for 3 units of fat, 7 units of protein, and 9 units of carbs in each meal. Suppose an individual has three possible foods to choose from to meet these requirements. Each ounce of the food contains

Food 1: 3 units of fat, 4 units of protein, 1 unit of carbs

Food 2: 2 units of fat, 5 units of protein, 3 units of carbs

Food 3: 4 units of fat, 1 unit of protein, 2 units of carbs

Let x , y , and z denote the number of ounces of the first, second, and third foods that a person will consume at the main meal. Find a linear system in x , y , and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

fat	$3x + 2y + 4z = 3$
protein	$4x + 5y + z = 7$
carb	$x + 3y + 2z = 9$